Mathematics

4 ESO

esobook

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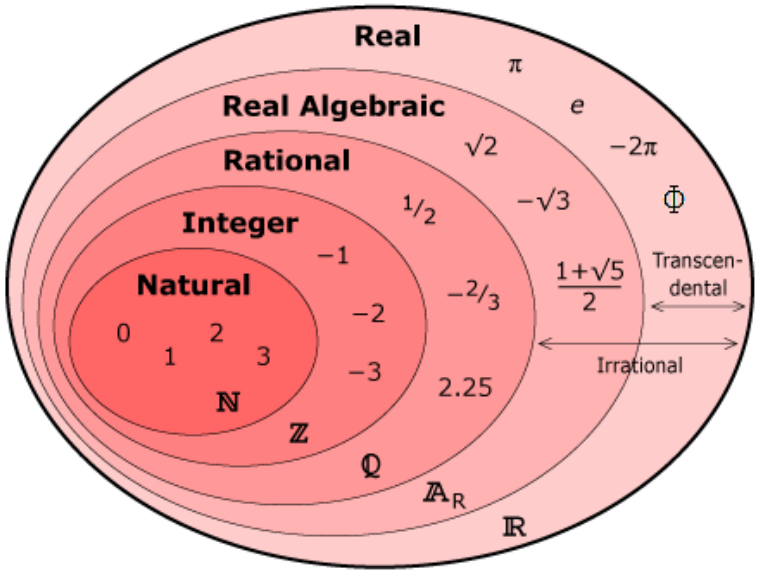
Chapter 1

REAL NUMBERS



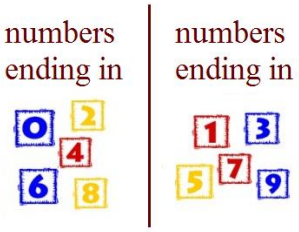
1.1 Classification

In mathematics, we classify numbers into groups.

- **Natural numbers** (N) are what we use to count objects one by one. We can count students, tables…
- The **integers** (sometimes called "whole numbers") (Z) have no decimals and require no fraction. They are the natural numbers and the negative numbers. Sometimes number 0 is not considered a natural number, but it always is a whole number (or integer).
- **Rational numbers** (Q) are those that we can express as a fraction: -1/4, 5, 0, 2/3, 4.1 …
- The **real algebraic numbers** (irrational, I) are those that we can't express as a fraction, for example: $\sqrt{10}$
- **Transcendental numbers** are special cases: they are real but not algebraic. For example π, Φ …

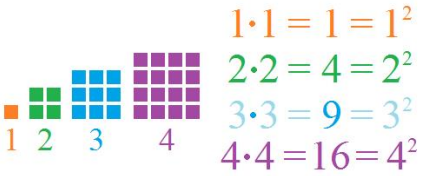
1 Which group of numbers would you use to measure temperature? Why?

Other categories



There are other classifications for numbers, for example:

- **Even** numbers are: 2, 4, 6, 8, … or -2, -4, -6, -8, …
- **Odd** numbers are: 1, 3, 5, 7, … or -1, -3, -5, -7, …
- **Prime** numbers are: 2, 3, 5, 7, 11, 13, 17, …
- **Square** numbers are: 1, 4, 9, 16, 25, 36, 49, 64, …



Some numbers are in more than one group. For example a prime number is also an integer.

The "smaller groups" belong also to the "larger groups". For example the integers are also rational numbers, although not all rational numbers are integers.

It is also very common to classify numbers into two categories:

- **Rational**
- **Irrational** (the real numbers that are not rational)

Make a table to classify (into the main categories) these numbers:
5/7 3.666… -2 5 8/2 2.123 0.123456…

	N	*Z*	*Q*	*I*
5/7	✗	✗	✓	✗
3.666…	✗	✗	✓	✗
-2	✗	✓	✓	✗
5	✓	✓	✓	✗
8/2	✓	✓	✓	✗
2.123	✗	✗	✓	✗
0.123456…	✗	✗	✗	✓

To what other categories do the numbers -2 and 5 belong to?
-2 is an **even** number and a **prime** number.
5 is an **odd** number and a **prime** number.

2 Make a table to classify (into the main categories) these numbers: 2 ½ e -3

3 Make a table to classify (into all categories) these numbers: 1.444… -9/3 π 0 1 1.12112112… -4/5 2.3333

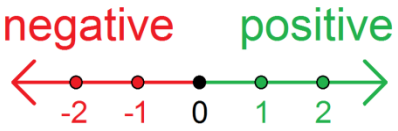
4 True or False?

a) An integer is a rational number.
b) A rational number is an integer.
c) A number is either rational or irrational, but not both.
d) An irrational number is also a real number.
e) All integers are natural numbers.
f) There is an integer that is not a rational number.

1.2 The number line

Density

Between two rational numbers on a number line, there is always another rational number. That property is called the **density** of the rational numbers.



Mark some fractions on a number line. No matter how dense the number line is, there is another number between the two numbers.

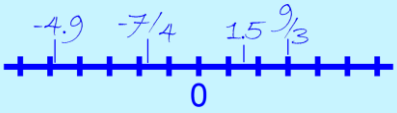
Example: 6/15 and 11/30, they are very close... but 23/60 is between them.

Absolute value

The absolute value of a number is the distance on the number line from 0 to the number. For that reason, the absolute value of a number is always a positive number or zero. We represent the absolute value of a number using vertical bars, ||.

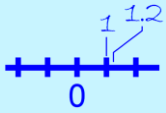
$|-3| = +3$ $|+5| = +5$ $|0| = 0$

➤ Represent these numbers on the number line: 1.5 ; -7/4 ; 9/3 ; -4.9



➤ Find two numbers between 7/9 and 1.4 and represent them on the number line.

There are infinite numbers between 7/9 and 1.4, for example 1 and 1.2



5 Represent these numbers on the number line:

a) 1.5
b) -6/2
c) 1/9

6 Find two fractions and two decimal numbers that are between 1/3 and 0.45 and represent them on the number line.

7 Represent the following numbers on a number line:

a) |4| c) |-5/2|
b) |-1| d) |11/2|

8 Find the absolute value of these numbers:

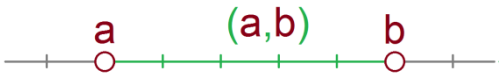
a) 6 b) – 9

Intervals

An interval is a set of real numbers between two points: a and b (a and b are the endpoints of the interval).

Open Interval

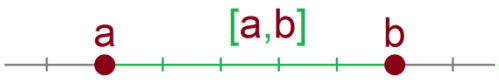
An open interval (a, b) is the set of real numbers greater than a, and smaller than b.



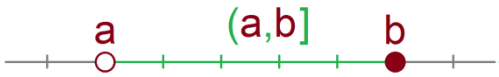
(a, b) = { x ∈ R ; a < x < b} {"any number that is real ; larger than a but smaller than b"}

Closed Interval

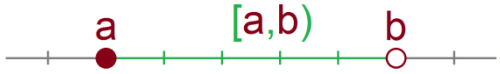
A closed interval [a, b] is the set of real numbers greater than or equal to a, and less than or equal to b.



[a, b] = {x ∈ R ; a ≤ x ≤ b} {"any number that is real ; equal to or larger than a and equal to or smaller than b"}



Half-Closed Intervals



(a, b] = {x ∈ R ; a < x ≤ b}
[a, b) = {x ∈ R ; a ≤ x < b}

Half-closed intervals are also called half-open intervals.

When there are a set of points formed by two or more of these intervals, we use the sign U (**Union**).

9 Express the following intervals in different ways:

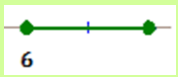
a) [-2,6)
b) { x ∈ R ; -3 ≤ x < 5}
c) [-3,-1]
d) (-6,4)
e) { x ∈ R ; 4 < x ≤ 8}
f) Real numbers bigger than or equal to 5 and smaller than 7
g) Real numbers smaller than 3 and bigger than or equal to 0
h) Real numbers between -3 and 3

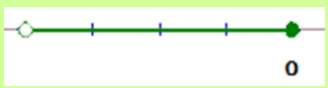
10 Represent the numbers that verify the following relations on the number line:

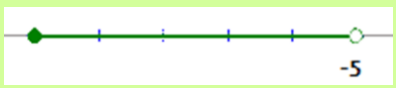
a) |x| < 1
b) |x| ≤ 1
c) |x| > 1
d) |x| ≥ 1

11 What kind of intervals are the following ones?

a)

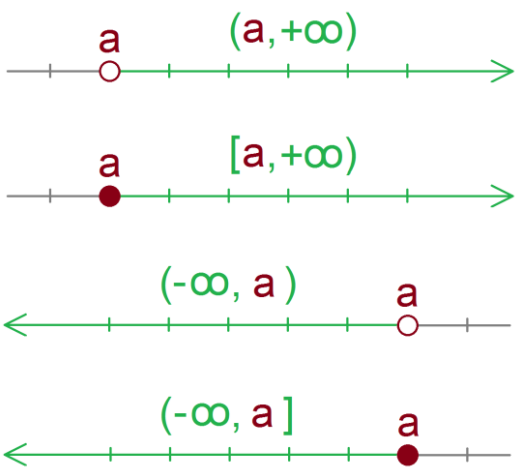
b) 

c) 

d) 

Half lines (Rays)

A "**half-line**" (also called "ray") are all the numbers in a number line from a point, a, (a number) to infinite or minus infinite.



Sometimes the ray contains (includes) the number. Sometimes it doesn't.

$a < x$ $\quad (a, +\infty) = \{x \epsilon R:\ a < x < \infty\}$

$a \leq x$ $\quad [a, +\infty) = \{x \epsilon R:\ a \leq x < \infty\}$

$x < a$ $\quad (-\infty, a) = \{x \epsilon R:\ -\infty < x < a\}$

$x \leq a$ $\quad (-\infty, a] = \{x \epsilon R:\ -\infty < x \leq a\}$

Balls

A ball is the space inside a sphere. In other words: it is all the points inside a sphere of radius r ($r > 0$) centered at point a.

An **open ball**, $B(a\ ;\ r)$, excludes the points on the sphere.

$$B(a\ ;\ r) = \{x \in R\ \ ;\ \ a-r < x < a+r\}$$

A **closed ball**, $B[a;\ r]$, includes the points on the sphere.

$$B[a\ ;\ r] = \{x \in R\ \ ;\ \ a-r \leq x \leq a+r\}$$

12 Represent the set $\{\ x \epsilon R\ \ ;\ \ |x-4| \leq 2\ \}$ using intervals.

13 Draw on the number line the following sets

a) $\{\ x \epsilon R\ ;\ |2x-6| > 5\ \}$

b) $\{\ x \epsilon R\ ;\ |-2x+7| < 2\ \}$

c) $\{\ x \epsilon R\ ;\ |x+4| \geq 8\ \}$

d) $\{\ x \epsilon R\ ;\ |4-x| \leq 10\ \}$

e) $\{\ x \epsilon R\ ;\ |3x-7| > 2\ \}$

f) $\{\ x \epsilon R\ ;\ |2-3x| < 3\ \}$

14 What is a "ball"?

15 Express these balls using intervals:

a) B[5,2] b) B(-2,3) c) B(0,4) d) B(-4,6)

1.3 Transcendental numbers

All irrational numbers have infinite non-recurrent decimal places, so we can't express them as a fraction. "Transcendental numbers" are special cases within the irrational numbers.

Π The best known irrational number is **π**, which is the relationship between the perimeter of the circle and its diameter.

$\pi = \frac{perimeter}{diameter}$ = 3.141592653589...

e Number **e** appears in growth processes, in radioactive decay and in the formula of the catenary, which is the curve that is found in power lines.

e = 2.718281828459...

Φ Artists use the **"golden ratio", Φ,** in the proportions of their works.

Φ = 1.6180339887…

1.4 Rounding errors

When we round a decimal number, we make an error. We can consider that error in two different ways (absolute or relative):

- The "**absolute error**" is the absolute value of the difference:

  E_a = | Number – Approximation |

- The "**relative error**" is a relation between the "size" of the error and the "size" of the number. We can calculate it with the formula:

  $E_r = (E_a / \text{Number}) * 100$

Round 26.398703 to two decimal places and calculate the absolute and relative errors:

We round: 26.40

$E_a = |26.398703 - 26.40|$

$E_a = 0.001297$

$E_r = 0.001297 / 26.398703 \cdot 100$

$E_r = 0.00491\%$

16 Round the following numbers to two decimal places and calculate the absolute and relative errors:

a) 2.56453
b) 63.32678
c) 0.9674
d) 5.2382
e) 9.999
f) 1.4634

17 Search on the internet to find what the "golden ratio" is. Define it in one line and give an example of where you would use it.

Activities

18 Which of these numbers are rational? π, 3.3, 6/4, $\sqrt{5}$, 0.

19 Classify the following numbers:

a) $-\sqrt{9}$
b) $-6/3$
c) 2.121212
d) 0
e) $-4.343434\ldots$
f) $\sqrt{6}$
g) $(-3)^2$
h) $-\sqrt{8/2}$

20 Rememeber: A fraction can be expressed as a decimal number: we divide numerator by denominator:

3 : 5 = **0.6**

Convert the following rational numbers into decimals:

a) 2/5
b) ¼
c) -7/4
d) 2/3
e) -4/5
f) 7/11

21 Convert the following numbers into decimals. Say which type of decimal number each number is:

a) 1/5
b) 6/4
c) 2/32
d) 11/5
e) 7/3
f) 56/11
g) 243/432
h) 231/47

22 Rememeber: To convert a decimal number to a fraction there is a set of methods that we saw in 3rd ESO, but it is not always possible to convert a decimal number to fraction.

Convert these decimal numbers into fractions:

a) 2.75
b) 4.6666…
c) $1.\widehat{56}$
d) 4.25252525…
e) 1.26666…
f) 0.52525252…

23 Calculate:

a) $1.\overline{36} + 8.\overline{25}$
b) $4.\overline{5} + 6.\overline{7}$
c) $3.\overline{46} + 4.2\overline{95}$
d) $1.25 \cdot 2.\overline{5}$
e) $0.0\overline{3} : 2.9\overline{2}$

24 Calculate:

a) $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{18}$

b) $\frac{2}{7} - 3\left(\frac{4}{5} + \frac{2}{9}\right) + 16$

c) $\frac{2}{3} - 4\left(\frac{5}{3} - \frac{6}{4}\right) - 3 \cdot \frac{5}{6}$

d) $3(-2) - \frac{1}{3}\left(\frac{1}{2} - \frac{1}{4}\right)$

e) $6 - \frac{3}{4} : \frac{9}{2} - 2\left(1 - \frac{1}{2}\right)$

f) $\frac{2}{5} - \frac{2}{5}\left(1 - \frac{1}{2}\right) + \frac{3}{4}\left(\frac{7}{3} - \frac{9}{6}\right)$

g) $\left(1 - \frac{1}{2}\right)\left(\frac{1}{3} - 1\right)\left(1 - \frac{1}{5}\right) - 1$

25 Add the decimal numbers then convert to fraction.

a) 0.125 + 0.333333333…
b) 0.166666666… + 0.066666666…

26 On the previous activity, convert the decimal numbers to fractions, then add the fractions. Check that the result is the same.

27 Draw a number line. Represent 1.3 ; -4 and $\sqrt{7}$ on the number line.

28 Represent the following numbers on a number line:

a) |-2|
b) |0|
c) $|-\sqrt{9}|$
d) |-1|
e) |-3/2|
f) |+5|
g) |-1.222…|
h) |+2|

29 Represent the following numbers on a number line:

a) π
b) e
c) 1 / |-4/2|
d) $\sqrt[3]{-8}$

30 On the number line, draw these intervals:

a) $6 < x < 10$
b) $\sqrt{7} < x$
c) $-7 \leq x \leq -3$
f) $-2 < x \leq 4$
g) $-2 > x$
h) $-2 < x$

31 On the number line, draw these intervals:

a) $-5 \leq x \leq 2$
b) $\sqrt{5} \leq x \leq 2$
i) $0 < x$
i) $0 < x^2$

32 On the number line, draw these intervals:

a) $[0, 3]$
b) $(-4 , 7]$
c) $(0 , 8)$
d) $(-\infty , 6]$
e) $[-2 , 4)$
f) $(-\infty , 2)$
g) $[-3 , 5]$
h) $[3 , \infty)$

33 Represent these inequalities on the number line:

a) $|x| > 5$
b) $|x| \leq 1$
c) $|2x| > 6$
d) $|x - 1| > 4$
e) $|x| < 4$
f) $|x| \geq 0$
g) $|3x| < 4$
h) $|1 - 2x| < 2$

34 Using intervals, represent the following sets:

a) $\{x \epsilon R \ ; \ |x + 1| \leq 2\}$
b) $\{x \epsilon R \ ; \ |x - 5| > 3\}$
c) B(-2,5)
d) B[1,5]
e) $\{x \epsilon R \ ; \ |2 - x| \leq 3\}$
f) $\{x \epsilon R \ ; \ |3x + 4| > 2\}$
g) $\{x \epsilon R \ ; \ |2x - 6| \leq 5\}$
h) $\{x \epsilon R \ ; \ |2 - 3x| > 4\}$
i) B(0,1)

35 Draw the following sets on the number line:

a) $\{x \epsilon R \ ; \ |x - 1| > 5\}$
b) $\{x \epsilon R \ ; \ |2x + 4| < 8\}$
c) $\left\{x \epsilon R \ ; \ |x - 5| \geq \frac{1}{2}\right\}$
d) $\{x \epsilon R \ ; \ |-4 + 2x| \leq 2\}$
e) $\{x \epsilon R \ ; \ |x + 8| > 1\}$
f) $\left\{x \epsilon R \ ; \ \left|\frac{2x}{3} - 3\right| > 3\right\}$
j) $\left\{x \epsilon R \ ; \ \left|1 - \frac{4x}{5}\right| < 2\right\}$
g) $\left\{x \epsilon R \ ; \ \left|\frac{2}{5} - 3x\right| \geq 6\right\}$

36 Round the following numbers to two decimal places and to three decimal places:

a) 2.56453
b) 63.32678
c) 0.9674
d) 5.2382
e) 9.999
f) 1.4634

37 Round the following numbers to two decimal places and find the absolute and relative errors in each case:

a) 2.546
b) 4.373
c) 5.765
d) 4.376
e) 5.637
f) 4.764

38 Round the following numbers to three decimal places and find the absolute and relative errors:

a) 6.15746
b) 8.2537
c) 0.4582
d) 54.5376
e) 0.0638
f) 1.698

39 Round the following numbers to ten decimal places and find the absolute and relative errors:

a) 2.5537646473945
b) 2856.375376658
c) 234.23879344781240
d) 5.637826870001
e) 0.00000048975214
f) 1.6981111111111156

Word Problems

40 A classroom is 6 m wide and 8 m long. What type of number is its diagonal?

41 Which group of numbers would you use to count students? Why?

42 What is the "density" of the rational numbers?

43 The sides of four equilateral triangles are: $\sqrt[4]{128}$; $\frac{1}{0.26}$; π ; e
Order them from smaller to greater.

44 A = -2 ; B = -1 ; C = 1 and D = 3
Find:

a) Two natural numbers between A and D.
b) Two integers between A and C.
c) Three rational numbers, not integers, between B and C.
d) Three irrational numbers between C and D.

45 Find three numbers that are between 0 and 24/25 and represent them on the number line.

46 Write down three irrational numbers between 1.34 and 1.36.

47 Write down three rational numbers between -2.34 and -2.28. Find each fraction in its lowest terms.

48 Express each of the statements as an interval:

a) Numbers greater than -1 and smaller or equal than 5
b) Numbers between 0 and 2
c) Numbers bigger than -4
d) Numbers smaller or equal than 3
e) Numbers greater or equal than 4 and smaller than 5
f) Numbers greater or equal than -1

49 Express the following intervals in different ways:

a) (-2,2)
b) $\{x\epsilon R: -1 \leq x < 1\}$
c) Numbers bigger than -1 and smaller or equal than 0
d) $\{x\epsilon R: 0 < x \leq 10\}$
e) [-1,6)
f) Numbers smaller than 3 and bigger than 0
g) Numbers between -2 and 64
h) (-1, +∞)

50 A bus driver has travelled 176.6 km in the first part of a trip, 87.765 km in the second part, and 162.68 km in the third part.

How many more kilometres must the driver travel if the total trip is 990 km?
Give your answer using only one decimal place and find the absolute and relative errors.

51 From a water bottle, we take 184.565 cl of water, then 128.784 cl and finally 84.455 cl.

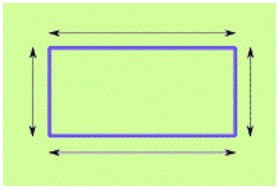
At the end, there are 0.60 litres in the bottle.
How much water was in the bottle originally?
Give your answer with two decimal places and find the absolute and relative errors.

52 In a certain year, city A consumed 2/3 and city B consumed 1/6 of all the electricity consumed in the country.

If all the other cities combined consumed 18 million KWh that year:

a. How many KWh did city B consume that year?
b. How many KWh did the rest of the country consume that year?

53 Find the perimeter of a rectangle with a length of 7/8 cm and a width of 2/5 cm. Round your answer to one decimal place.



54 In March, Mark had 157.32 € in his bank account.

In April, he put 50 €, 75.48 € and 25.62 € in the account.

In October, he took 10.85 € and 111.93 € from the account.

a) What is the final balance?
b) Round your answer to the units.
c) Round your answer to one decimal place and find the absolute error.
d) Round your answer to one decimal place and find the relative error.

55 A tenth of the students wear glasses. A fourth of the students that wear glasses sometimes wear contact lenses.

a) What fraction of the students sometimes wears contact lenses?
b) If there is a total of 420 students, how many students wear glasses?

56 John borrowed some money from Tom. John must pay 3/2 of what he borrowed minus 50 euros. He paid Tom a total of 118 euros.

How much money did John borrow?

57 A family must travel 5,200 miles in total. After travelling 4/13 of the distance, they have ten more days to complete the trip.

How many miles/day should they travel for the remainder of their trip?

58 Π is approximately 22/7

If we use that approximation to calculate the circumference of a circle with radius 5 m:

a. What is the absolute error?
b. What is the relative error?

59 True or False?

a) The natural numbers contain the real numbers.
b) The sum of two irrational numbers is always an irrational number.
c) The square root of a fraction is an irrational number.
d) Every decimal number can be written as a fraction.
e) An irrational number can be an integer.
f) A rational number is a decimal number.
g) Φ is the symbol for the "golden ratio", it is a "transcendental number".